A Numerical Investigation of an Adaptive Power Series Approximation to Compartmental Models

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Introduction

In mathematics, compartmental modeling is a framework often used when the members of a set can be grouped into distinct categories or compartments. When the resulting system is coupled in a nonlinear way, numerical solutions are often the only way to approximate the true solutions. Here, we use power series solutions to represent the solution in each compartment. We show the development of the SIR model, used typically to simulate infectious diseases, in which the function $S(t)$ represents the number of susceptible individuals, the function $I(t)$ represents the number of infected individuals, and the function $R(t)$ represents the number of recovered individuals in a given population. We consider these quantities as parts of the whole fixed population such that the derivatives of these fractions are the basis of our model.

We follow this by calculating the equilibrium values to support our model, as well as the series solutions. In order to gauge the accuracy of our model, we identify the true solution as the one calculated by utilizing Euler's method. We compare the true solutions and the series solutions using multiple intervals over an extended time frame.

The project goal is to model the spread of an infectious disease by utilizing dynamical systems. Dynamical systems are made up of differential equations relating unknowns that depend on time. Considering a fixed population, N, the percentages that are incorporated in our model are as follows:

$$s(t) = \frac{S(t)}{N}, \quad i(t) = \frac{I(t)}{N}, \quad \text{and} \quad r(t) = \frac{R(t)}{N}.$$

Our model consists of the first-order coupled (nonlinear) differential equations for s, i, and r.

The change in s and i depends on the rate at which an infected person spreads the disease, where β represents the significance of the interaction between s and i. The change in both i and r depends on the recovery rate, γ, and the population size, i. The change in r and s depends on the length of time that a recovered individual becomes susceptible again. This is governed by the parameter η. Note this also has an impact on the rate of change in s. In particular, the model we consider is

$$\frac{ds}{dt} = -\beta si + \eta r, \quad \frac{di}{dt} = \beta si - \gamma i, \quad \text{and} \quad \frac{dr}{dt} = \gamma i - \eta r.$$

Equilibrium Values

We consider the question, at what point does time not play as critical of a role? In order to analyze this, we begin with a look at the stability of the constant, or equilibrium solutions, to the dynamical system.

The equilibrium values [2] of the system are the values for which $\boldsymbol{f}(\boldsymbol{v}) = \boldsymbol{0}$. For our SIR system, we obtain, aside from the origin $(0,0,0)$, and the disease free equilibrium solution, $\boldsymbol{e}_{df} = (1,0,0)$,

$$\boldsymbol{e}_{eq} = (e_s, e_i, e_r) = \left(\frac{\gamma}{\beta}, \frac{\gamma(\beta-\gamma)}{\beta(\eta+\gamma)}, \frac{\eta(\beta-\gamma)}{\beta(\eta+\gamma)}\right).$$

Series Solutions

The aim is to generate power series series solutions for each of s, i, and r and to develop a recurrence relation for the coefficients in order to determine what role time plays in the equation. We assume the series have the following form:

$$s(t) = \sum_{n=0}^{\infty} s_n t^n, \quad i(t) = \sum_{n=0}^{\infty} i_n t^n, \quad \text{and} \quad r(t) = \sum_{n=0}^{\infty} r_n t^n.$$

Inserting these equations into the derivatives yields the following recurrence relations for the coefficients for $n \geq 0$:

$$s_{n+1} = \frac{1}{n+1}\left(-\beta \sum_{p=0}^{n} s_{n-p} i_p\right) + \eta r_n$$

$$i_{n+1} = \frac{1}{n+1}\left(\beta \sum_{p=0}^{n} s_{n-p} i_p\right) + \gamma i_n$$

$$r_{n+1} = \frac{1}{n+1}\left(\gamma i_n - \eta r_n\right).$$

As is suggested in [1], practical implementation requires a restart (or recentering) of the series and we have to use only a finite number of terms. Consequently, we set

$$s(t, t', N) = \sum_{n=0}^{N} s_{t',n}(t-t')^n,$$

and similarly define $i(t, t', N)$ and $r(t, t', N)$.

The choice of distance between the t'-value is central to the success of the implementation. We want to determine a radius of convergence for each of the series representations on intervals whose minimum value is t'. We denote this interval by

$$\mathcal{I}_{t',\varepsilon} = [t', t' + \varepsilon].$$

Given the lack of explicit formulae for the coefficients and noting that not all terms are positive, we consider the series of positive terms, $\sum_{n=0}^{N} |s_{t',n}|(t-t')^n$. and seek values of c and b for which $|s_{t',n}| \sim ce^{bn}$.

Given suitable values for c and b, we note the ratio

$$\left|\frac{s_{t',n+1}}{s_{t',n}}\right| \approx \left|\frac{ce^{b(n+1)}(t-t')^{n+1}}{ce^{bn}(t-t')^n}\right| = \mathrm{e}^b|t-t'|$$

indicates the series will converge for an ε value generally less than e^{-b}.

Adaptive Algorithm - fixed N-value:

(0) Choose a truncation value, N, for the series, determine a method for choosing the length of each subinterval, ε in $\mathcal{I}_{t',\varepsilon}$, and initialize $t' = t_0$.

Then, until $t' \geq t_{end}$, repeat steps $(1) - (4)$:

(1) Compute coefficients in the recurrence relations with initial conditions $s(t'), i(t')$, **and** $r(t')$.

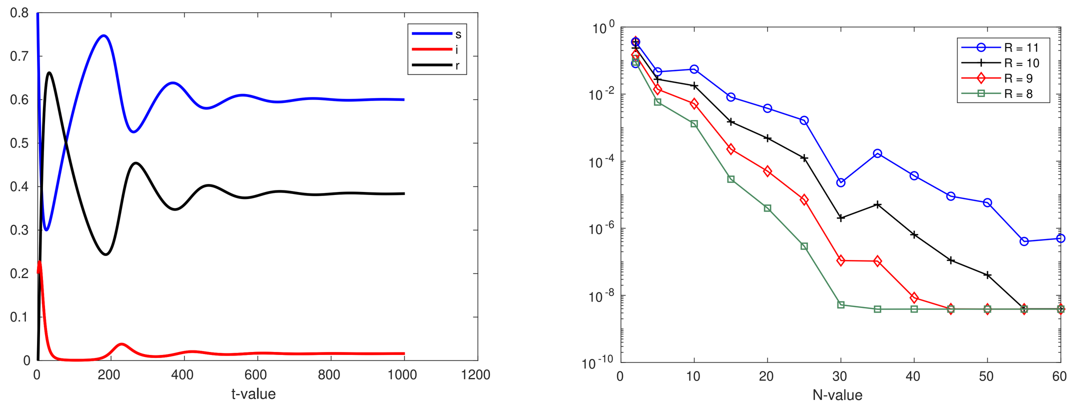
(2) Compute ε for $\mathcal{I}_{t',\varepsilon}$.

(3) Compute the values of $s(t, t', N), i(t, t', N)$ **, and** $r(t, t', N)$ **for** $t \in \mathcal{T} \cap \mathcal{I}_{t',\varepsilon}$

(4) Set t' the right end point of $\mathcal{I}_{t',\varepsilon}$ in (3).

Numerical Example

In this example, we demonstrate the implementation of the algorithm without the adaptive choice of R in step (3) using fixed values of N and R. We continue with the parameters set in the previous example and consider a time frame from $t = 0$ to $t = 1000$. The algorithm is run for R-values of $11, 10, 9$, and 8, and for N-values beginning with $2, 5$ and then in increments of 5 to 60. The results for $R = 8$ and $N = 45$ are shown in part (a) of the figure below. The convergence of the error is shown in part (b) of the figure below where the results are intuitive in the sense that the error decreases as both R decreases and N increases.



(a) $R = 8$ and $N = 45$ (b) $E(N, R)$

The N-value has more influence on the accuracy in computing the equilibrium values for smaller t-values, as shown in the table below.

Time	N	Relative Error: s t = 1000	Relative Error: s t = 2000	Relative Error: i t = = 1000	Relative Error: i t = 2000	Relative Error: r t = 1000	Relative Error: r t = 2000
R=13	5	3.675 e−4	8.970 e−8	5.597 e−3	5.568 e−6	8.074 e−4	1.170 e−6
	40	1.443 e−4	8.416 e−7	4.769 e−3	1.925 e−6	2.681 e−5	1.235 e−6
R=12	5	2.608 e−5	8.937 e−8	5.473 e−3	4.838 e−6	6.356 e−4	1.195 e−6
	40	3.611 e−6	8.802 e−8	5.102 e−3	2.789 e−6	1.562 e−4	1.259 e−6
R=6	5	3.137 e−5	8.815 e−7	5.115 e−3	2.825 e−6	1.641 e−4	1.260 e−6
	40	3.224 e−5	8.818 e−7	5.112 e−3	2.818 e−6	1.626 e−4	1.259 e−6

Conclusion

We have presented an algorithm to implement series approximations that is adaptive in the R. Using the adaptive algorithm, we were able to obtain accurate results at a reduced computational cost. By quantifying the way the coefficients decayed, we approximated the radius of convergence.

A more thorough investigation to quantify this relationship is needed. This would lead to an algorithm that is not only adaptive in the R, but also in the number of terms that is needed. Also, in practice, parameters are not constant, so it would be interesting to see how the system reacts when parameters are variable.

References

[1] J.J. Nieto H.M. Srivastava, I. Area. Power-series solution of compartmental epidemiological models. *Mathematical Biosciences and Engineering*, 18(4):3274–3290, 2021.

[2] Lawrence Perko. *Differential equations and dynamical systems*, volume 7 of *Texts in Applied Mathematics*. Springer-Verlag, New York, third edition, 2001.