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Brahmagupta quadrilaterals are lattice quadrilaterals

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A Brahmagupta quadrilateral is a cyclic quadrilateral with integer side lengths, integer diagonals, and integer area. We show that every Brahmagupta quadrilateral can be placed in the xy-plane so that its vertices have integer coordinates, generalizing a similar result for Heronian triangles (triangles with integer side lengths and integer area).

1. Introduction

It was shown by Paul Yiu [2001] that *every Heronian triangle is a lattice triangle*. A Heronian triangle is a triangle with integer side lengths and integer area, and a polygon is lattice if it can be placed in the xy-plane so that its vertices have integer coordinates. In this paper, we generalize Yiu's result to a special kind of quadrilateral, known as a *Brahmagupta quadrilateral*.

2. Brahmagupta quadrilaterals

Brahmagupta quadrilaterals are named after the Indian mathematician, Brahmagupta, who lived from approximately 598 to 670.

Definition. A *Brahmagupta quadrilateral* is a cyclic quadrilateral with integer side lengths, integer diagonals, and integer area.

For a polygon to be cyclic, all of its vertices lie on the same circle, as seen in Figure 1, which shows a Brahmagupta quadrilateral with side lengths a, b, c, and d, diagonals e and f, and area K.

Brahmagupta quadrilaterals satisfy many nice properties because they are cyclic. For example, notice that the angles $\theta_1 = \angle ADB$ and $\theta_2 = \angle ACB$ are equal because both are inscribed angles subtended by the same chord, a.

For any cyclic quadrilateral, we can apply Brahmagupta's area formula,

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

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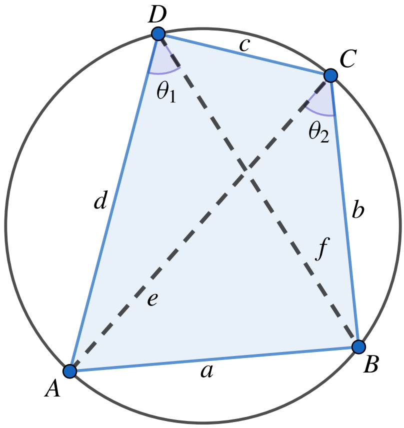

Figure 1. A Brahmagupta quadrilateral.

where s is the semiperimeter of the quadrilateral. This is a generalization of Heron's area formula for triangles, which can be deduced from Brahmagupta's formula by setting $d = 0$:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}. \tag{1}$$

We can also utilize Ptolemy's theorem, which relates the diagonals of a cyclic quadrilateral to its side lengths:

$$ef = ac + bd. \tag{2}$$

3. Strategy

Our goal is to show that Brahmagupta quadrilaterals are lattice, so we need to find a placement in the xy-plane such that the vertices of the quadrilateral have integer coordinates. We'll use a strategy developed by Jan Fricke to prove that Heronian triangles are lattice triangles in a different way than Yiu did. We first look for a placement where the coordinates of the vertices are rational instead of integer and then use the following theorem to find an integer placement.

Theorem 1 (Fricke's theorem [2001]). *Given a finite set of points with rational coordinates in the xy-plane* (*including the origin*) *for which the distance between any two points is an integer*, *there exists a single rotation of the points to integer coordinates.*

The proof of Fricke's theorem uses the number theory of *Gaussian integers*, complex numbers $x + iy$, where x and y are integers. Fricke's theorem was generalized to three-dimensions in [Marshall and Perlis 2013], using the number theory of quaternions. It was also generalized in [Knopf et al. 2016] to lattices in the xy-plane other than $\mathbb{Z}$, using the number theory of quadratic fields.

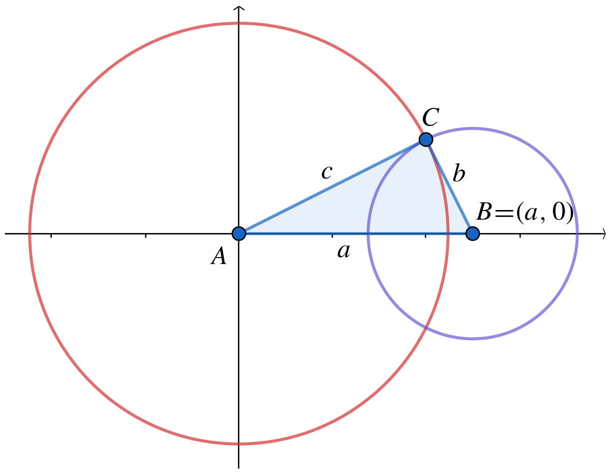

Figure 2. Rational placement of a Heronian triangle.

To see how Fricke used Theorem 1 to prove that Heronian triangles are lattice triangles, we let T be a Heronian triangle with side lengths a, b, and c, and area Δ. First, we place one vertex, A, at the origin and another vertex, B, on the x-axis to represent the side length, a. We then use intersections of circles to obtain the third vertex C, as shown in Figure 2. The equations of the circles are given by

$$x^2 + y^2 = c^2,$$
$$(x-a)^2 + y^2 = b^2.$$

Solving this system and simplifying using Heron's area formula (1) gives the coordinates

$$A = (0,0), \quad B = (a,0), \quad C = \left(\frac{c^2 - b^2 + a^2}{2a}, \frac{2\Delta}{a}\right). \tag{3}$$

The coordinates are rational since a, b, c, and Δ are integers. We now have a set of points in the xy-plane that satisfies the hypotheses of Fricke's theorem since the side lengths of the triangle are the distances between the points. Thus, we can find a rotation to move the triangle to an integer placement, proving that T is a lattice triangle.

4. Placements of Brahmagupta quadrilaterals

Let Q be a quadrilateral (not necessarily Brahmagupta) with side lengths a, b, c, and d, diagonals e and f, and area K. To find a placement of Q, we notice that the diagonals of a quadrilateral form two *diagonal triangles*, as shown in Figure 3. One triangle has side lengths a, b, e, and the other triangle has side lengths a, d, f, respectively.

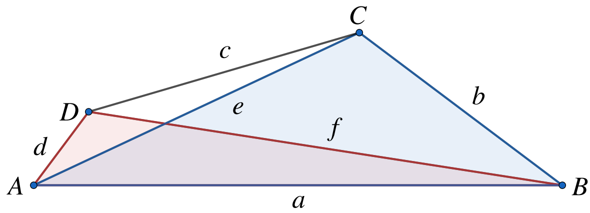

Figure 3. The diagonal triangles of a quadrilateral.

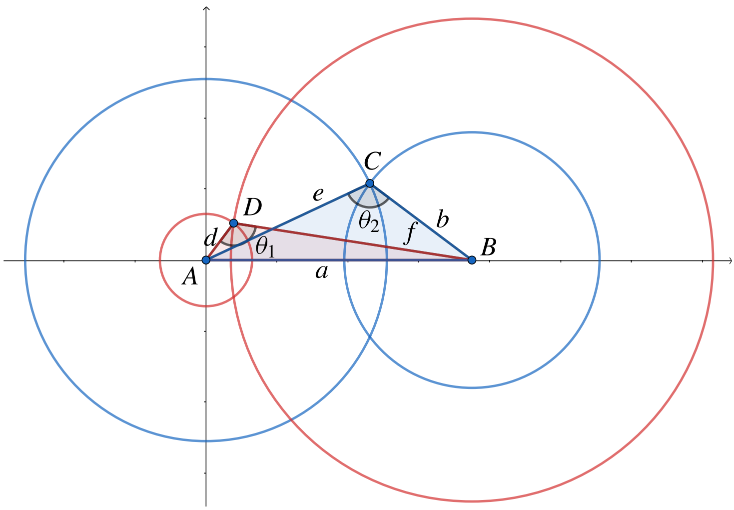

Figure 4. Placement of the diagonal triangles.

We place each diagonal triangle separately, using the formula for triangle coordinates (3) derived in the previous section. This gives the coordinates

$$\begin{aligned} A &= (0,0), \quad & C &= \left(\frac{e^2 - b^2 + a^2}{2a}, \frac{2\Delta_{abe}}{a} \right), \\ B &= (a,0), \quad & D &= \left(\frac{d^2 - f^2 + a^2}{2a}, \frac{2\Delta_{adf}}{a} \right), \end{aligned} \tag{4}$$

where Δ_{abe} and Δ_{adf} are the areas of the diagonal triangles.

We first want to verify that (4) gives us our original quadrilateral, Q, when we connect the vertices C and D in Figure 4. We can either use the quadrilateral congruence theorem SASAS to see this or more familiar triangle congruence theorems as follows. We note that $\angle DAB = \angle DAC + \angle CAB$. Thus, d, $\angle DAC$, and e must be congruent in any two placements, forcing the distance between C and D to be the same by the triangle congruence theorem SAS. In the case of cyclic quadrilaterals (which include Brahmagupta quadrilaterals), we can also directly compute the distance between C and D.

Theorem 2. *Let Q be a cyclic quadrilateral with side lengths a, b, c, and d, diagonals e and f, and area K. Then, the distance between C and D is equal to c, where C and D are the coordinates given in (4).*

Proof. Suppose we are given a cyclic quadrilateral, Q, with side lengths a, b, c, and d, diagonals e and f, and area K. Let

$$C = \left(\frac{e^2 - b^2 + a^2}{2a}, \frac{2\Delta_{abe}}{a}\right), \quad D = \left(\frac{d^2 - f^2 + a^2}{2a}, \frac{2\Delta_{adf}}{a}\right),$$

where Δ_{abe} and Δ_{adf} are the areas of the diagonal triangles. Recall that the distance between $C = (x_1, y_1)$ and $D = (x_2, y_2)$ is given by the distance formula:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We want to show that the distance between C and D is c or equivalently,

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = c^2.$$

We'll first rewrite $(x_1 - x_2)^2$ using Heron's area formula and the law of cosines. Substituting the x-values from the C and D coordinates into the distance formula, we obtain

$$\begin{aligned}(x_1 - x_2)^2 &= \left(\frac{e^2 - b^2 + a^2}{2a} - \frac{d^2 - f^2 + a^2}{2a}\right)^2 \\ &= \frac{(e^2 - b^2 + a^2)^2 - 2(e^2 - b^2 + a^2)(d^2 - f^2 + a^2) + (d^2 - f^2 + a^2)^2}{4a^2}.\end{aligned}$$

Manipulating Heron's formula (1), we find that the areas of the diagonal triangles satisfy the equations

$$\begin{aligned}(4\Delta_{abe})^2 &= -a^4 - b^4 - e^4 + 2a^2b^2 + 2a^2e^2 + 2b^2e^2, \\ (4\Delta_{adf})^2 &= -a^4 - d^4 - f^4 + 2a^2d^2 + 2a^2f^2 + 2d^2f^2.\end{aligned}$$

An algebraic computation will then verify that

$$(e^2 - b^2 + a^2)^2 = 4a^2e^2 - (4\Delta_{abe})^2, \tag{5}$$

$$(d^2 - f^2 + a^2)^2 = 4a^2d^2 - (4\Delta_{adf})^2. \tag{6}$$

By the law of cosines, we know that

$$\begin{aligned}a^2 &= b^2 + e^2 - 2be\cos\theta_2, \\ a^2 &= d^2 + f^2 - 2df\cos\theta_1,\end{aligned}$$

which implies

$$(a^2 - b^2 - e^2)(a^2 - d^2 - f^2) = 4bdef\cos\theta_1\cos\theta_2.$$

A bit more algebra then yields

$$-2(e^2-b^2+a^2)(d^2-f^2+a^2) \\ = 4e^2f^2-4a^2e^2+4b^2d^2-4a^2d^2-8bdef\cos\theta_1\cos\theta_2. \quad (7)$$

Adding (5), (6), and (7), and simplifying therefore gives us

$$(x_1-x_2)^2 = \frac{-(4\Delta_{abe})^2+4e^2f^2+4b^2d^2-8bdef\cos\theta_1\cos\theta_2-(4\Delta_{adf})^2}{4a^2}. \quad (8)$$

We'll second rewrite $(y_1-y_2)^2$ using the trigonometric area formulas:

$$\Delta_{abe} = \tfrac{1}{2}be\sin\theta_2, \qquad \Delta_{adf} = \tfrac{1}{2}df\sin\theta_1.$$

Substituting the y-values from the C and D coordinates and using the above area formulas, we obtain

$$\begin{aligned}(y_1-y_2)^2 &= \left(\frac{2\Delta_{abe}}{a}-\frac{2\Delta_{adf}}{a}\right)^2 \\ &= \frac{(4\Delta_{abe})^2-32\Delta_{abe}\Delta_{adf}+(4\Delta_{adf})^2}{4a^2} \\ &= \frac{(4\Delta_{abe})^2-8bdef\sin\theta_1\sin\theta_2+(4\Delta_{adf})^2}{4a^2}. \qquad (9)\end{aligned}$$

Finally, we'll add (8) and (9) and simplify to find that

$$(x_1-x_2)^2+(y_1-y_2)^2 = \frac{e^2f^2+b^2d^2-2bdef(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)}{a^2}.$$

To further simplify this, we'll use trigonometric identities:

$$\begin{aligned}&\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2 \\ &\quad = \tfrac{1}{2}(\cos(\theta_1-\theta_2)+\cos(\theta_1+\theta_2))+\tfrac{1}{2}(\cos(\theta_1-\theta_2)-\cos(\theta_1+\theta_2)) \\ &\quad = \cos(\theta_1-\theta_2).\end{aligned}$$

Therefore,

$$(x_1-x_2)^2+(y_1-y_2)^2 = \frac{e^2f^2+b^2d^2-2bdef\cos(\theta_1-\theta_2)}{a^2}.$$

We note that, up to this point, we have not used the fact that Q is cyclic. Recall that $\theta_1=\theta_2$ in cyclic quadrilaterals, so that

$$(x_1-x_2)^2+(y_1-y_2)^2 = \frac{b^2d^2+e^2f^2-2bdef}{a^2} = \left(\frac{ef-bd}{a}\right)^2.$$

By Ptolemy's theorem (2), we know $ef=ac+bd$. Thus, $c=(ef-bd)/a$ and

$$(x_1-x_2)^2+(y_1-y_2)^2=c^2,$$

proving that the distance between C and D is equal to c. □

We now assume that Q is a Brahmagupta quadrilateral. Recall that our strategy is to find a rational placement of Q and then apply Theorem 1 (Fricke's theorem) to find an integer placement. Thus, we need to verify all the coordinates in (4) are rational when Q is a Brahmagupta quadrilateral. Since a, b, d, e, and f are integers, all of the coordinates except the y-coordinates of C and D are immediately seen to be rational. We know the area of the entire quadrilateral is an integer, but we don't yet have information about the areas of the diagonal triangles, Δ_{abe} and Δ_{adf}.

Theorem 3. *The diagonal triangles are Heronian in any Brahmagupta quadrilateral.*

Proof. Let Q be a Brahmagupta quadrilateral with side lengths a, b, c, and d, and diagonals e and f. Each diagonal of the quadrilateral splits the quadrilateral into two triangles: (a, b, e) and (c, d, e) from diagonal e and (a, d, f) and (b, c, f) from diagonal f. (See Figure 1.) All side lengths are integers so it remains to show that the areas Δ_{abe}, Δ_{cde}, Δ_{adf}, and Δ_{bcf} are also integer.

Buchholz and MacDougall [2008] studied cyclic polygons with rational side lengths and rational area. (We note that their definitions of Heron(-ian) triangle and Brahmagupta quadrilateral are a little different from our definitions.) Translating their results to our situation gives the desired result.

Lemma 2 of [Buchholz and MacDougall 2008] tells us that if a cyclic quadrilateral with rational side lengths and rational area has one rational diagonal, then both diagonals must be rational and all four diagonal triangles have rational area. Thus, the areas Δ_{abe}, Δ_{cde}, Δ_{adf}, and Δ_{bcf} are rational.

Theorem 1 of [Buchholz and MacDougall 2008] tells us that if a triangle has integer side lengths and rational area, then it must have integer area. Thus, the areas Δ_{abe}, Δ_{cde}, Δ_{adf}, and Δ_{bcf} are integer, and the diagonal triangles of Q are Heronian. □

Theorem 3 guarantees that all the coordinates in our placement of Q are rational. Now, we can apply Theorem 1 (Fricke's theorem) to rotate the rational placement to an integer placement. This gives our main result.

Theorem 4. *Every Brahmagupta quadrilateral is a lattice quadrilateral.*

5. Finding integer placements of Brahmagupta quadrilaterals

The proof of Fricke's theorem (Theorem 1) provides a method to find the rotation that gives the integer placement of a Brahmagupta quadrilateral. We'll show how the process works through an example.

Let Q be the Brahmagupta quadrilateral with side lengths $a = 377$, $b = 260$, $c = 145$, $d = 152$; diagonals $e = 273$, $f = 345$; and area $K = 43470$. We first find a rational placement of Q using the coordinates given in (4). We use Heron's

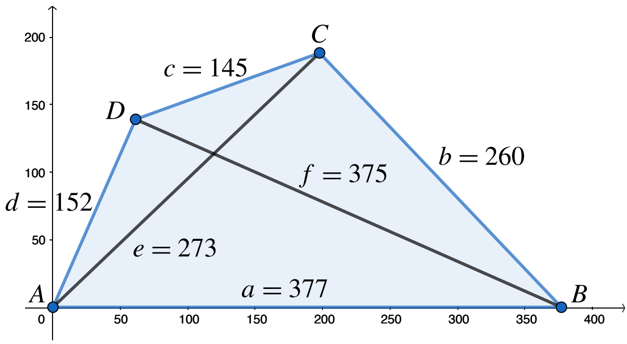

Figure 5. Rational placement of Q.

formula (1) to find the areas of the diagonal triangles, Δ_{abe} and Δ_{adf}. The rational placement, shown in Figure 5, is given by

$$A = (0, 0), \qquad C = \left(\tfrac{5733}{29}, \tfrac{5460}{29}\right),$$
$$B = (377, 0), \quad D = \left(\tfrac{23104}{377}, \tfrac{52440}{377}\right).$$

We want to find a single rotation that will eliminate the denominators from the coordinates. We'll instead show how to find separate rotations to remove each prime factor of the denominators one at a time. In our example, the denominators are 29 and $377 = 29 \cdot 13$. So, the prime factors we must consider are $p_1 = 29$ and $p_2 = 13$. Then, the single rotation would be given by a composition of these rotations.

To find the rotations attached to each prime, we'll think of points in the xy-plane as complex numbers:

$$(x, y) \leftrightsquigarrow x + iy.$$

Then, rotations can be described as multiplication by another complex number, $a + bi$, where $a^2 + b^2 = 1$.

To find the rotation attached to $p_1 = 29$, we write 29 as a sum of squares, $29 = 5^2 + 2^2$, allowing us to factor 29 into Gaussian integers: $29 = (5+2i)(5-2i)$. The proof of Fricke's theorem guarantees that one of the following complex numbers gives the rotation that removes the prime 29 from the denominators:

$$\frac{5-2i}{5+2i} = \frac{21}{29} - \frac{20}{29}i \quad \text{or} \quad \frac{5+2i}{5-2i} = \frac{21}{29} + \frac{20}{29}i.$$

The correct rotation turns out to be

$$\frac{5-2i}{5+2i} = \frac{21}{29} - \frac{20}{29}i.$$

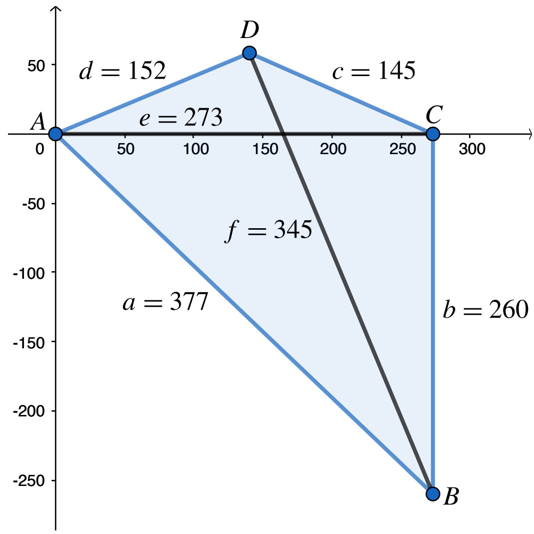

Figure 6. Rotated placement of Q eliminating 29 from denominators.

After we multiply each of our rational coordinates (thought of as complex numbers) by $\frac{21}{29} - \frac{20}{29}i$, our coordinates are

$$A = (0, 0), \qquad C = (273, 0),$$
$$B = (273, -260), \quad D = \left(\tfrac{1824}{13}, \tfrac{760}{13}\right).$$

Figure 6 shows the rotated quadrilateral.

To find the rotation attached to $p_2 = 13$, we write 13 as a sum of squares, $13 = 3^2 + 2^2$, allowing us to factor 13 into Gaussian integers: $13 = (3+2i)(3-2i)$.

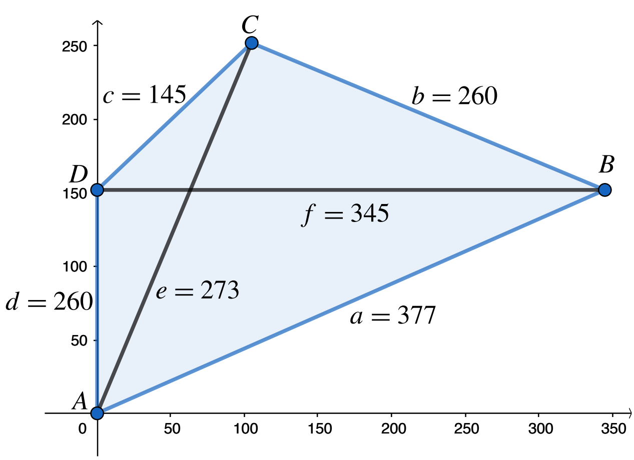

Figure 7. Integer placement of Q.

The proof of Fricke's theorem guarantees that one of the following complex numbers gives the rotation that removes the prime 13 from the denominators:

$$\frac{3-2i}{3+2i} = \frac{5}{13} - \frac{12}{13}i \quad \text{or} \quad \frac{3+2i}{3-2i} = \frac{5}{13} + \frac{12}{13}i.$$

The correct rotation turns out to be

$$\frac{3+2i}{3-2i} = \frac{5}{13} + \frac{12}{13}i.$$

After we multiply each of our rotated coordinates (thought of as complex numbers) by $\frac{5}{13} + \frac{12}{13}i$, our coordinates are

$$A = (0, 0), \qquad C = (105, 252),$$
$$B = (345, 152), \quad D = (0, 152).$$

This gives us our integer placement of Q, shown in Figure 7.

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